Geometry, Correlated Equilibria and Zero-Sum Games

Yannick Viossat

ECOLE POLYTECHNIQUE

CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Geometry, Correlated Equilibria and Zero-Sum Games

Yannick Viossat

Décembre 2003

Cahier n° 2003-032

LABORATOIRE D'ECONOMETRIE

1rue Descartes F-75005 Paris

(33) 1 55558215

http://ceco.polytechnique.fr/

mailto:labecox@poly.polytechnique.fr

Geometry, Correlated Equilibria and Zero-Sum Games

Yannick Viossat[1]

Décembre 2003

Cahier n° 2003-032

Résumé: Ce papier porte à la fois sur la géométrie des équilibres de Nash et des équilibres corrélés et sur une généralisation des jeux à sommes nulles fondée sur les équilibres corrélés. L'ensemble des distributions d'équilibres corrélés de n'importe quel jeu fini est un polytope, qui contient les équilibres de Nash. Je caractérise la classe des jeux tels que ce polytope (s'il ne se réduit pas à un singleton) contienne un équilibre de Nash dans son intérieur relatif. Bien que cette classe de jeux ne soit pas définie par une propriété d'antagonisme entre les joueurs, je montre qu'elle inclut et qu'elle généralise la classe des jeux à deux joueurs et à somme nulle.

Abstract: This paper is concerned both with the comparative geometry of Nash and correlated equilibria, and with a generalization of zero-sum games based on correlated equilibria. The set of correlated equilibrium distributions of any finite game in strategic form is a polytope, which contains the Nash equilibria. I characterize the class of games such that this polytope (if not a singleton) contains a Nash equilibrium in its relative interior. This class of games, though not defined by some antagonistic property, is shown to include and generalize two-player zero-sum games.

Mots clés : Equilibres corrélés, jeux à somme nulle, géométrie

Key Words : Correlated equilibria, zero-sum games, geometry.

Classification JEL: C72

[1] Laboratoire d'Econométrie, CNRS et Ecole polytechnique

1 Introduction

The correlated equilibrium concept (Aumann, [2]) generalizes the Nash equilibrium concept to situations where players may condition their behavior on payoff-irrelevant observations made before play[1]. Aumann showed that correlated equilibria are sometimes more efficient or more reasonable than Nash equilibria [2], and that playing a correlated equilibrium is the natural expression of Bayesian rationality [3]. The correlated equilibrium concept is also well suited to the study of biological conflicts in which the agents may have different "roles" [6] and has been implicitly used in theoretical biology ever since Maynard Smith and Parker [16].

The geometry of correlated equilibria is relatively simple. Indeed, the set of correlated equilibrium distributions of any finite game is a polytope and existence of correlated equilibria can actually be proved by linear programming [14]. It follows that, when the entries of the payoff matrices are rational, a correlated equilibrium with, say, maximum payoff-sum may be computed in polynomial time [10]. In sharp contrast, the set of Nash equilibria of a finite game may be disconnected, its connected components need not be convex, and computing a Nash equilibrium with maximum payoff-sum is NP hard, even in two-player games [10].

In the last decade, the comparative geometry of Nash and correlated equilibria has been further investigated. It has been found that, in two-player games, extreme Nash equilibria are extreme points of the polytope of correlated equilibrium distributions ([7], [11]), which we denote by C. More recently, Nau et al [19] showed that in any n*-player* game G, *all* Nash equilibria belong to the relative boundary of C, unless G satisfies a rather restrictive condition. More precisely, let us say that a pure strategy is *coherent* if it has positive probability in some correlated equilibrium distribution. Nau et al [19] showed that if a Nash equilibrium lies in the relative interior of C then G satisfies the following condition: *in any correlated equilibrium distribution, all the incentive constraints stipulating that a player has no incentive to "deviate" to a coherent strategy are binding*[2] (condition A).

This shows that the class of games with a Nash equilibrium in the relative interior of C is "small" but do not provide a precise characterization of this class of games. My first result is such a characterization. More precisely, let us call "prebinding" the games that satisfy the above condition A. I show that C contains a Nash equilibrium in its relative interior if and only if G is prebinding and C is not a singleton.

My second result is that, though they are not defined by requiring antagonism between the players, prebinding games include and generalize two-player zero-sum games. For instance, in two-player prebinding games, Nash equilibria are exchangeable and any correlated equilibrium payoff is a Nash equilibrium payoff. Prebinding games actually appear to be the first generalization of two-player zero-sum games whose definition is entirely based on correlated equilibria.

This paper is thus at the intersection of two literatures: the literature that studies the geometry of Nash and correlated equilibria and the literature that studies classes of two-player games which, in some sense, generalize zero-sum games (e.g. [1], [9],

[1] A formal definition of correlated equilibrium distributions will be given in the next section.
[2] A more formal statement and a proof of this result will be given in section 3.

[15], [17], [21]). Moreover, many proofs are based on dual reduction ([18], [23]): a technique which, to my knowledge, has never been applied. This paper thus also shows how dual reduction may be used to investigate the geometry of correlated equilibria.

The remaining of the paper is organized as follow: the next section is devoted to basic notations and definitions. In section 3, we define two classes of games: "binding" and "prebinding" games. The link between these two classes of games is studied in section 4. The class of games with a Nash equilibrium in the relative interior of the correlated equilibrium polytope is characterized in section 5. The last section shows that two-player prebinding games generalize two-player zero-sum games. Finally, elements of dual reduction are recalled in appendix A.

2 Notations

The analysis in this paper is restricted to finite games in strategic forms. Let $G = \{I, (S_i)_{i\in I}, (u_i)_{i\in I}\}$ denote a finite game in strategic form: I is the nonempty finite set of players, S_i the nonempty finite set of pure strategies of player i and $u_i : \times_{i\in I} S_i \to \mathbb{R}$ the utility function of player i. The set of (pure) strategy profiles is $S = \times_{i\in I} S_i$; the set of strategy profiles for the players other than i is $S_{-i} = \times_{j\in I-i} S_j$. Pure strategies of player i (resp. strategy profiles; strategy profiles of the players other than i) are denoted s_i or t_i (resp. s; s_{-i}). Similarly, mixed strategy of player i (resp. mixed strategy profiles, mixed strategy profiles of the players other than i) are denoted σ_i or τ_i (resp. σ; σ_{-i}). Thus, we may write (t_i, s_{-i}) (resp. (τ_i, σ_{-i})) to denote the strategy (resp. mixed strategy) profile that differs from s (resp. σ) only in that its $i-$component is t_i (resp. τ_i). For any finite set Σ, $\Delta(\Sigma)$ denotes the set of probability distributions over Σ. Finally, N denotes the cardinal of S.

2.1 Correlated equilibrium distribution

The set $\Delta(S)$ of probability distributions over S is an $N-1$ dimensional simplex, henceforth called *the simplex*. A *correlated strategy* of the players in I is an element of the simplex. Thus $\mu = (\mu(s))_{s\in S}$ is a correlated strategy if:

$$\text{(nonnegativity constraints)} \qquad \mu(s) \geq 0 \quad \forall s \in S \tag{1}$$

$$\text{(normalization constraint)} \qquad \sum_{s\in S} \mu(s) = 1 \tag{2}$$

For $(i, s_i, t_i) \in I \times S_i \times S_i$, let h_{s_i,t_i} denote the linear form on $\mathbb{R}^S$ which maps $x = (x(s))_{s\in S}$ to

$$h_{s_i,t_i}(x) = \sum_{s_{-i}\in S_{-i}} x(s)[u_i(s) - u_i(t_i, s_{-i})]$$

A correlated strategy μ is a *correlated equilibrium distribution* [2] (abbreviated occasionally in c.e.d.) if:

$$\text{(incentive constraints)} \qquad h_{s_i,t_i}(\mu) \geq 0 \quad \forall i \in I, \forall s_i \in S_i, \forall t_i \in S_i \tag{3}$$

Let $\mu \in \Delta(S)$, $i \in I$ and $s_i \in S_i$. If s_i has positive probability in μ, let $\mu(.|s_i) \in \Delta(S_{-i})$ be the correlated strategy given s_i of the players other than i:

$$\forall s_{-i} \in S_{-i}, \mu(s_{-i}|s_i) = \frac{\mu(s)}{\mu(s_i \times S_{-i})} \text{ where } \mu(s_i \times S_{-i}) = \sum_{s_{-i} \in S_{-i}} \mu(s)$$

The incentive constraints (3) mean that, for any player i and any pure strategy s_i of player i, either s_i has zero probability in μ (in which case $h_{s_i,t_i}(\mu) = 0$ for all t_i in S_i) or s_i is a best response to $\mu(.|s_i)$. A possible interpretation is as follow: assume that before play a mediator ("Nature", some device,...) chooses a strategy profile s with probability $\mu(s)$ and privately "recommends" s_i to player i. In this framework, the incentive constraints (3) stipulate that if all the players but i follow the recommendations of the mediator, then player i has no incentives to deviate from s_i to some other strategy t_i.

Since conditions (1), (2) and (3) are all linear, the set of correlated equilibrium distributions is a polytope, which we denote by C.

Notations and vocabulary: Let $s_i \in S_i$, $s \in S$ and $\mu \in \Delta(S)$. The strategy s_i (resp. strategy profile s) is *played* in the correlated strategy μ if $\mu(s_i \times S_{-i}) > 0$ (resp. $\mu(s) > 0$). Furthermore, the average payoff of player i in μ is

$$u_i(\mu) = \sum_{s \in S} \mu(s) u_i(s)$$

3 Definitions and remarks

3.1 Binding Games

Definition 3.1 *A game is binding if in any correlated equilibrium distribution all the incentive constraints are binding. Formally,*

$$\forall \mu \in C, \forall i \in I, \forall s_i \in S_i, \forall t_i \in S_i, h_{s_i,t_i}(\mu) = 0 \tag{4}$$

Let i be in I and s_i, t_i in S_i. Following Myerson [18], let us say that t_i *jeopardizes* s_i if $h_{s_i,t_i}(\mu) = 0$ for all μ in C. That is, if whenever s_i is played in a correlated equilibrium distribution μ, t_i is an alternative best response to $\mu(.|s_i)$. The concept of jeopardization is at the heart of the theory of dual reduction [18], [23]. Dual reduction, in turn, will be a key-tool to prove some of the main results of this article. It is thus useful to rephrase definition 3.1 in terms of jeopardization:

Alternate definition 3.2 *A game is binding if for all i in I any pure strategy of player i jeopardizes all his pure strategies.*

(Indeed the above condition is exactly:

$$\forall i \in I, \forall t_i \in S_i, \forall s_i \in S_i, \forall \mu \in C, h_{s_i,t_i}(\mu) = 0$$

which is equivalent to (4))

Example 3.3

$$G_1 = \begin{pmatrix} 1,-1 & 0,0 \\ 0,0 & 1,-1 \end{pmatrix} \qquad G_2 = \begin{pmatrix} 1,-1 & 0,0 & 0,-1 \\ 0,0 & 1,-1 & 0,-1 \end{pmatrix}$$

The game G_1 (i.e. Matching Pennies) is binding. Indeed, G_1 has a unique correlated equilibrium distribution: the Nash equilibrium σ in which both players play $(1/2, 1/2)$. Therefore, definition 3.1 boils down to: G_1 is binding if, in σ, all incentive constraints are binding. But σ is a completely mixed Nash equilibrium. Therefore, in σ, all incentive constraints are indeed binding and definition 3.1 is checked.

In contrast, G_2 is not binding. Indeed, there is still a unique correlated equilibrium distribution: the Nash equilibrium σ in which the row player plays $(\frac{1}{2}, \frac{1}{2})$ and the column player $(\frac{1}{2}, \frac{1}{2}, 0)$. But against σ_1, player 2 has a strict incentive not to play her third strategy.

3.2 Prebinding Games

Following Nau et al [19], let us define a strategy to be coherent if it is played in some correlated equilibrium. Formally,

Definition 3.4 *Let i be in I and s_i in S_i. The strategy s_i is coherent if there exists a correlated equilibrium distribution μ such that $\mu(s_i \times S_{-i})$ is positive.*

We denote by S_i^c the set of coherent strategies of player i. We can now define prebinding games:

Definition 3.5 *A game is* prebinding *if in any correlated equilibrium distribution all the incentive constraints stipulating not to "deviate" to a coherent strategy are binding. That is,*

$$\forall \mu \in C, \forall i \in I, \forall s_i \in S_i, \forall t_i \in S_i^c, h_{s_i,t_i}(\mu) = 0$$

Alternate definition 3.6 *A game is prebinding if every coherent strategy of every player jeopardizes all his other pure strategies.*

(Definitions 3.5 and 3.6 are equivalent, just as definitions 3.1 and 3.2). Note that if s_i is not coherent, then $h_{s_i,t_i}(\mu) = 0$ for all μ in C and all t_i in S_i. Therefore, definition 3.5 boils down to:

Alternate definition 3.7 *A game is* prebinding *if:*

$$\forall \mu \in C, \forall i \in I, \forall s_i \in S_i^c, \forall t_i \in S_i^c, h_{s_i,t_i}(\mu) = 0$$

Example 3.8 *Any game with a unique correlated equilibrium distribution is prebinding. For instance, the games G_1 and G_2 of example 3.3 are prebinding.*

Indeed, if G has a unique correlated equilibrium distribution σ, then σ is necessarily a Nash equilibrium. Furthermore, the set of coherent strategies of player i is simply the support of σ_i. Therefore, definition 3.7 boils down to: for any player i in I and any pure strategies s_i and t_i in the support of σ_i, $u_i(s_i, \sigma_{-i}) = u_i(t_i, \sigma_{-i})$. This condition is satisfied since σ is a Nash equilibrium. Therefore G is prebinding.

Example 3.9 *Any two-player zero-sum game is prebinding (see section 6 for a proof).*

For an example of a three-player binding and prebinding game, in which, moreover, extreme Nash equilibria are not extreme correlated equilibria, see Nau et al [19].

3.3 Remarks

First, in the definitions of binding and prebinding games, the utility functions only intervene via the best-response correspondences, so that:

Remark 3.10 *If G is binding (resp. prebinding) then any game that is best-response equivalent* [3]*[21] to G is binding (resp. prebinding).*

Second, there is a difference between a *correlated equilibrium* and a *correlated equilibrium distribution* [4,5]. We chose to phrase definitions 3.1 and 3.5 in terms of correlated equilibrium distributions. Equivalently, we could have defined binding and prebinding games in terms of correlated equilibria. For instance, the reader may check that: a game is binding if and only if in all correlated equilibria, all incentive constraints are binding[6].

4 Links between binding and prebinding games

In this section we study the link between binding and prebinding games. In so doing, we establish a lemma which will prove crucial to the next section. We first need to introduce the game G^c obtained from G by restricting the players to their coherent strategies:

$$G^c = \{I, (S_i^c)_{i\in I}, (u_i)_{i\in I}\}^7$$

For instance, in example 3.3, $G_2^c = G_1$ and $G_1^c = G_1$. We denote by $S^c = \times_{i\in I} S_i^c$ the set of strategy profiles of G^c and by $C^c \subset \Delta(S^c)$ the set of correlated equilibrium distributions of G^c. Since any correlated equilibrium distribution of G has support in S^c, the set of correlated equilibrium distributions of G may be seen as a subset of $\Delta(S^c)$. We then have:

[3] Two games with the same sets of players and strategies are *best-response equivalent* [21] if they have the same best-response correspondences.

[4] For all i in I, let M_i be a finite set, and let $M = \times_{i\in I} M_i$. Let $\nu \in \Delta(M)$. Consider the extended game in which: first, a point $m = (m_i)_{i\in I}$ is drawn at random according to the probability ν and m_i is privately announced to player i for all i; second, G is played (In this extended game, players can condition their behavior in G on their private information. A pure strategy of player i is thus a mapping from M_i to S_i.). A correlated equilibrium of G is a Nash equilibrium of such an extended game. A correlated equilibrium distribution is a probability distribution induced on S by some correlated equilibrium (this definition of c.e.d. may be shown to be equivalent to the one of section 2).

[5] Following Nau et al [19], I call C the *correlated equilibrium polytope*. This is abusive, since C is actually the polytope of correlated equilibrium *distributions*.

[6] In the sense that in all correlated equilibria, for any message m_i received by player i with positive probability, any strategy of player i is a best response to the conditional strategy of the other players given m_i.

[7] To be precise, the utility functions in G^c are the utility functions *induced* on $S^c = \times_{i\in I} S_i^c$ by the utility functions u_i of the original game.

Remark 4.1 *Any correlated equilibrium distribution of G is a correlated equilibrium distribution of G^c. That is, $C \subseteq C^c$. Furthermore, the inclusion may be strict.*

The first assertion is straightforward : if μ is in C, then $h_{s_i,t_i}(\mu) \geq 0$, for all i in I and all pure strategies s_i and t_i of player i. Therefore, a fortiori, $h_{s_i,t_i}(\mu) \geq 0$ for all i in I and all *coherent* pure strategies of player i; that is, μ is in C^c. The fact that the inclusion may be strict is less intuitive. Indeed, at first glance, it seems that eliminating strategies that are never played in correlated equilibria should not affect the set of correlated equilibrium distributions. But in the following example this intuition fails:

Example 4.2

	s_2	t_2
s_1	$1,1$	$0,1$
t_1	$0,1$	$1,0$

	s_2	t_2
s_1	$1,1$	$0,1$

Let G denote the left game. Then G^c is the game on the right[8]*. In both games Nash equilibrium and correlated equilibrium distributions coincide. In G^c any correlated strategy is, trivially, a Nash equilibrium distribution; in contrast, in G, a mixed strategy profile σ is a Nash equilibrium if and only if $\sigma_1(t_1) = 0$ and $\sigma_2(t_2) \leq 1/2$. Thus,*

$$C = \{\mu \in \Delta(S) : \mu(t_1 \times S_2) = 0 \text{ and } \mu(s) \geq \mu(s_1, t_2)\} \subsetneq C^c$$

Finally note that G is prebinding. Therefore the inclusion $C \subset C^c$ may be strict even if we restrict our attention to prebinding games.

We now link binding and prebinding games:

Proposition 4.3 *(a) A game G is prebinding if and only if G^c is binding; (b) a game is binding if and only if it is prebinding and every pure strategy of every player is coherent.*

We first need a lemma:

Lemma 4.4 *(a) If G is binding, then G has a completely mixed Nash equilibrium. (b) If G is prebinding, then G has a Nash equilibrium σ such that: σ has support S^c; in σ, all players have a strict incentive not to deviate from coherent to incoherent strategies. Formally,*

$$\forall s \in S^c, \sigma(s) > 0 \tag{5}$$

$$\forall i \in I, \forall s_i \in S_i^c, \forall t_i \in S_i - S_i^c, h_{s_i,t_i}(\sigma) > 0 \tag{6}$$

(For prebinding games, condition (6) may be rephrased as follow: for every player i and every pure strategy s_i of player i, s_i is a best response to σ_{-i} if and only if s_i is coherent.)

[8]The strategy t_1 cannot be played in a c.e.d. for the following reason: if $\mu(t_1, t_2) > 0$ then $u_2(\mu)$ is less than 1, i.e. less than what s_2 guarantees, hence μ cannot be an equilibrium. But if $\mu(t_1, t_2) = 0$ then player 1 cannot be incited to play t_1.

Proof. We only prove the second assertion. The proof of the first assertion is similar and simpler. The proof is based on dual reduction. The reader unfamiliar with dual reduction is advised to first go through appendix A.

(i) Define $g(\alpha, s)$ as in equation (12) of appendix A. By [20, proposition 2] and by convexity of the set of dual vectors there exists a dual vector α such that:

$$\forall s \in S, [\mu(s) = 0 \text{ for all } \mu \text{ in } C \Rightarrow g(\alpha, s) > 0] \tag{7}$$

We may assume α full (otherwise, take a strictly convex combination of α and some full dual vector).

(ii) In the full dual reduction induced by α, as in all full dual reductions, all strategies of $S_i - S_i^c$ are eliminated [23, proposition 5.13]; furthermore, since the coherent strategies of player i jeopardize each other, they must either all be eliminated or all be grouped together (see [23, section 4]); since some strategies of player i must remain in the reduced game, the first possibility is ruled out; therefore, all coherent strategies of player i are grouped in a single mixed strategy σ_i, with support S_i^c. In the reduced game, the resulting strategy profile $\sigma = (\sigma_i)_{i \in I}$ is the only strategy profile, hence trivially a Nash equilibrium. By [23, proposition 5.7], this implies that σ is a Nash equilibrium of G. Moreover, σ has support S^c.

(iii) Let $i \in I$ and let s_i (resp. t_i) be a coherent (resp. incoherent) pure strategy of player i. Let $\tau = (t_i, \sigma_{-i}) \in \Delta(S)$. Since t_i is incoherent, $\mu(t) = 0$ for all μ in C and all t_{-i} in S_{-i}. Therefore, by (7),

$$\sum_{t_{-i} \in S_{-i}} \sigma_{-i}(t_{-i}) g(\alpha, t) > 0$$

Since σ_j is α_j-invariant for all $j \neq i$, the above boils down to:

$$u_i(\alpha_i * t_i, \sigma_{-i}) - u_i(t_i, \sigma_{-i}) > 0$$

Therefore t_i is not a best response to σ_{-i}. Since σ is a Nash equilibrium and $\sigma_i(s_i) > 0$, s_i is a strictly better response than t_i to σ_{-i}. As $\sigma_i(s_i) > 0$ this implies $h_{s_i,t_i}(\sigma) > 0$. ■

By lemma 4.4, binding games have a completely mixed Nash equilibrium, hence:

Corollary 4.5 *If G is binding, then every pure strategy of every player is coherent. That is, $G = G^c$.*

We can now prove proposition 4.3:

Proof of (a): The game G^c is binding if and only if

$$\forall \mu \in C^c, h_{s_i,t_i}(\mu) = 0 \qquad \forall i \in I, \forall s_i \in S_i^c, \forall t_i \in S_i^c \tag{8}$$

Similarly, by definition 3.7, G is prebinding if and only if

$$\forall \mu \in C, h_{s_i,t_i}(\mu) = 0 \qquad \forall i \in I, \forall s_i \in S_i^c, \forall t_i \in S_i^c \tag{9}$$

Since $C \subset C^c$ (remark 4.1), (8) implies (9). [9] We show that (9) implies (8) by contraposition. Assume that (8) does not hold. Then:

$$\exists \mu \in C^c, \exists i \in I, \exists s_i \in S_i^c, \exists t_i \in S_i^c, h_{s_i,t_i}(\mu) > 0$$

By lemma 4.4, there exists μ^* checking (6). For $\epsilon > 0$ small enough, $\mu_\epsilon = \epsilon\mu + (1-\epsilon)\mu^*$ is in C. But $h_{s_i,t_i}(\mu_\epsilon) > 0$. This contradicts (9).

Proof of (b): Assume G binding. By corollary 4.5, $G = G^c$. Therefore G^c is binding. Therefore, by proposition 4.3 (a), G is prebinding. Grouping these observations: $G = G^c$ and G is prebinding. Conversely, assume that (i) $G = G^c$ and (ii) G is prebinding. By (ii) and proposition 4.3 (a), G^c is binding. Therefore, by (i), G is binding. ■

5 The Geometry of Nash and Correlated Equilibria

Nau et al [19] proved the following:

Proposition 5.1 *If G has a Nash equilibrium σ in the relative interior of C, then:*[10]

(a) The Nash equilibrium σ assigns positive probability to every coherent strategy of every player; that is, σ has support S^c.

(b) G is prebinding.[11]

Proof. If (a) is not checked, then σ satisfies with equality some nonnegativity constraint which is not satisfied with equality by all correlated equilibrium distributions, hence σ belongs to the relative boundary of C. Assuming now that condition (a) is checked, σ renders indifferent every player among its coherent strategies; therefore σ satisfies with equality all incentive constraints of type $h_{s_i,t_i}(.) \geq 0$, where s_i and t_i are coherent. If G is not prebinding, at least one of these constraints is not satisfied with equality by all correlated equilibrium distributions, hence σ belongs to the relative boundary of C. ■

The aim of this section is to prove a converse of this result. Namely,

Proposition 5.2 *If a game is prebinding, then either C is a singleton or C contains a Nash equilibrium in its relative interior.*

Proposition 5.2, together with example 3.8 and proposition 5.1, allows to characterize prebinding games:

Theorem 5.3 *A game G is prebinding if and only if C is a singleton or C contains a Nash equilibrium in its relative interior. Thus, C contains a Nash equilibrium in its relative interior if and only if G is prebinding and C is not a singleton.*

[9] Example 4.2 shows that the implication $(9) \Rightarrow (8)$ is not as trivial.

[10] We abusively identify here and in what follows a Nash equilibrium and the independent distribution it induces on $\Delta(S)$.

[11] The term "prebinding" is mine: Nau et al write that G satisfies the property of definition 3.5.

Note that C is a singleton if and only if its relative interior is empty. So theorem 5.3 could be rephrased as follow: a game is prebinding if and only if the relative interior of C is empty or contains a Nash equilibrium.

Proofs Theorem 5.3 is straightforward, so we only need to prove proposition 5.2. We first need a lemma:

Lemma 5.4 *Let G be prebinding and assume that C is not a singleton. A Nash equilibrium of G belongs to the relative interior of C if and only if it checks conditions (5) and (6) of lemma 4.4.*

Proof. Let σ be a Nash equilibrium of G. By lemma 4.4, G has a Nash equilibrium - hence a correlated equilibrium distribution - checking (5) and (6). Therefore, if σ does not check (5) or (6), there exists a nonnegativity or an incentive constraint which is binding in σ but not in all correlated equilibrium distributions; hence σ belongs to a strict face of C. Conversely, assume that σ checks (5) and (6). Note that there exists an neighborhood Ω of σ in $\mathbb{R}^S$ in which (5) and (6) are checked. Let E denote the set of points $x = (x(s))_{s\in S}$ of $\mathbb{R}^S$ such that: $\sum_{s\in S} x(s) = 1$ and $\forall i \in I, \forall s_i \in S_i, \forall t_i \in S_i^c, h_{s_i,t_i}(x) = 0$. Since G is prebinding, the affine span of C is a subset of E. Furthermore, $\Omega \cap E \subset C$. Finally, since C is not a singleton, E is not a singleton either. Therefore, σ belongs to the relative interior of C. ■

We can now prove proposition 5.2: assume that G is prebinding. By lemma 4.4(b), there exists a Nash equilibrium σ checking (5) and (6). If furthermore C is not a singleton, lemma 5.4 implies that σ belongs to the relative interior of C. ■

We end this section with two remarks on lemma 5.4: first, in binding games, condition (6) is void. Thus, the analogous of lemma 5.4 for binding games is: a Nash equilibrium of a binding game G belongs to the relative interior of C if and only if it is completely mixed and C is not a singleton; second, we might wonder whether, for prebinding games, condition (6) is really needed. That is, if G is prebinding, do all Nash equilibria with support S^c check condition (6) and thus belong to the relative interior of C ? The following example shows that this is not so.

Example 5.5 *Reconsider the game G of example 4.2. Let σ denote the Nash equilibrium of G given by $\sigma_1(s_1) = 1$ and $\sigma_2(s_2) = 1/2$; σ has support S^c (and thus belongs to the relative interior of C^c) but lies on the relative boundary of C.*

6 Two-player prebinding games

In this section we first show that two-player zero-sum games are prebinding but that a prebinding game need not be best-response equivalent to a zero-sum game. We then show that, nevertheless, some of the properties of the equilibria and equilibrium payoffs of zero-sum games extend to prebinding games. We then discuss the interest and implications of these findings.

6.1 Prebinding games and zero-sum games

Proposition 6.1 *A two-player game which is best-response equivalent to a zero-sum game is prebinding.*

Proof. In view of proposition 3.10 we only need to prove the result for two-player zero-sum games. So let G be a two-player zero-sum game and v its value. Note in succession that:

(i) In any c.e.d. the payoff for player 1 given a move is at least the value of the game. Formally,

$$\forall \mu \in C, \forall s_1 \in S_1, \mu(s_1 \times S_2) > 0 \Rightarrow \sum_{s_2 \in S_2} \mu(s_2|s_1) u_1(s) \geq v$$

(indeed, s_1 is a best-response to $\mu(.|s_1)$ and player 1 can guarantee v)

(ii) In any c.e.d. the average payoff for player 1 is the value of the game:

$$\forall \mu \in C, u_1(\mu) = v$$

(indeed, $u_1(\mu) \geq v$ by (1) and symmetrically $u_2(\mu) \geq -v$; but $u_2(\mu) = -u_1(\mu)$)

(iii) In any c.e.d., the payoff of player 1 given a move is the value of the game. Formally,

$$\forall \mu \in C, \forall s_1 \in S_1, \mu(s_1 \times S_2) > 0 \Rightarrow \sum_{s_2 \in S_2} \mu(s_2|s_1) u_1(s) = v$$

(use (i) and (ii))

(iv) For all μ in C and all s_1 in S_1, if $\mu(s_1 \times S_2) > 0$ then $\sigma_2 = \mu(.|s_1)$ is an optimal strategy of player 2.
(Otherwise $u_1(s_1, \sigma_2) = \sum_{s_2 \in S_2} \mu(s_2|s_1) u_1(s) > v$, since s_1 is a best response to σ_2. This contradicts (iii).)

(v) If a pure strategy t_1 of player 1 is coherent, then it is a best response to any optimal strategy of player 2.
(If t_1 is coherent there exists μ in C and s_2 in S_2 such that $\mu(t_1|s_2)$ is positive. Assume that there exists an optimal strategy σ_2 of player 2 to which t_1 is not a best response. By playing σ_2 against $\mu(.|s_2)$, player 2 would get strictly more than $-v$. Therefore $\mu(.|s_2)$ is not an optimal strategy of player 1. This contradicts the analogous of (iv) for player 2.) [12]

(vi) Let $s_1 \in S_1, t_1 \in S_1^c$. Then, for all μ in C, $h_{s_1,t_1}(\mu) = 0$
(if $\mu(s_1 \times S_2) = 0$, this holds trivially; otherwise $\mu(.|s_1)$ is optimal by (v); so by (iv), t_1 is an alternative best response to $\mu(.|s_1)$).

It follows from (vi) and from the symmetric of (vi) for player 2 that G is prebinding. ∎

The following example shows that the converse of proposition 6.1 is false. That is, a two-player prebinding game need not be best-response equivalent to a zero-sum game.

[12] (v) can also be proved directly by writing the maximization programm of player 1 and its dual. (v) then appears as a complementary slackness property.

Example 6.2 (Bernheim [5])

$$G = \begin{pmatrix} 0,7 & 2,5 & 7,0 \\ 5,2 & 3,3 & 5,2 \\ 7,0 & 2,5 & 0,7 \end{pmatrix}$$

This game is not best-response equivalent to a zero-sum game.[13] *However, G has a unique correlated equilibrium distribution (see [20, p.439] for a proof); hence, as a particular case of example 3.8, G is prebinding.*

We now show that, nevertheless, some of the main properties of two-player zero-sum games extend to prebinding games. Noticeably, in two-player prebinding games, the Nash equilibria are exchangeable and any correlated equilibrium payoff is a Nash equilibrium payoff.

6.2 Equilibria of prebinding games

Let us first introduce some notations: we denote by NE the set of Nash equilibria of G and by NE_i the set of Nash equilibrium strategies of player i. That is,

$$NE_i = \{\sigma_i \in \Delta(S_i), \exists \sigma_{-i} \in \times_{j \in I-i} \Delta(S_j), (\sigma_i, \sigma_{-i}) \in NE\}$$

Our first result is that:

Proposition 6.3 *In a two-player prebinding game:*

(a) *NE_1 and NE_2 are convex polytopes.*

(b) *$NE = NE_1 \times NE_2$. That is, the Nash equilibria are exchangeable.*

We first need a lemma:

Lemma 6.4 *Let G be a two-player prebinding game and let $\sigma_1 \in \Delta(S_1)$ be a mixed strategy of player 1. The following assertions are equivalent:*

(i) *σ_1 is a Nash equilibrium strategy. That is, $\sigma_1 \in NE_1$.*

(ii) *For some pure strategy s_2 of player 2, σ_1 is the conditional strategy of player 1 given s_2 in some correlated equilibrium distribution. Formally, $\exists \mu \in C, \exists s_2 \in S_2, \mu(s_2 \times S_1) > 0$ and $\sigma_1 = \mu(.|s_2)$.*

[13] Indeed, assume by contradiction that G is best-response equivalent to a zero-sum game. Exploiting the symmetries of the game, it is possible to show that G is also best-response equivalent to a zero-sum game G' with payoffs for player 1:

$$\begin{pmatrix} -\alpha & -\beta & \alpha \\ \beta & 0 & \beta \\ \alpha & -\beta & -\alpha \end{pmatrix}$$

for some real numbers α and β. Furthermore, in G, the two first strategies of player 1 are both best responses to $(1/5, 1/5, 3/5)$ and to $(0, 2/3, 1/3)$. Since G and G' are best-response equivalent, this must also be the case in G'. This implies $\alpha = \beta = 0$. Therefore, in G', any strategy of player 1 is a best-response to the first strategy of player 2. But this is not the case in G: a contradiction.

(iii) Every pure strategy played in σ_1 is coherent and all coherent strategies of player 2 are best responses to σ_1.

(The symmetric results for σ_2 in $\Delta(S_2)$ hold obviously just as well.)

Proof. (i) trivially implies (ii) and (ii) implies (iii) by definition 3.5. So we only need to prove that (iii) implies (i). Let σ_1 check (iii) and let $\tau_2 \in NE_2$. Necessarily, any pure strategy played in τ_2 is coherent. Since any coherent strategy of player 2 is a best response to σ_1, τ_2 is a best response to σ_1. Similarly, by the analogous of $(i) \Rightarrow (iii)$ for player 2, any coherent strategy of player 1 is a best response to τ_2. Since all pure strategies played in σ_1 are coherent, σ_1 is a best response to τ_2. Grouping these results, we get that (σ_1, τ_2) is a Nash equilibrium, hence $\sigma_1 \in NE_1$. ■

We now prove proposition 6.3: it follows from the proof of lemma 6.4 that if $\sigma_1 \in NE_1$, then for any $\tau_2 \in NE_2$, (σ_1, τ_2) is a Nash equilibrium. This implies that Nash equilibria are exchangeable (point (b)). Furthermore, from the equivalence of (i) and (iii) it follows that NE_1 can be defined by a finite number of linear inequalities. Therefore, NE_1 is a polytope, and so is NE_2 by symmetry (point (a)). ■

Our second result is that if μ is a correlated equilibrium distribution, then the product of its marginals is a Nash equilibrium. More precisely:

Proposition 6.5 *Let μ be a correlated equilibrium distribution of a two-player prebinding game. Let $\sigma_1 \in \Delta(S_1)$ (resp. $\sigma_2 \in \Delta(S_2)$) denote the marginal probability distribution of μ on S_1 (resp. S_2). That is, $\forall s_1 \in S_1, \sigma_1(s_1) = \mu(s_1 \times S_2)$. Let $\sigma = (\sigma_1, \sigma_2)$ so that σ is the product of the marginals of μ. We have:*

(a) σ is a Nash equilibrium

(b) The average payoff of the players is the same in σ and in μ. That is, $\forall i \in \{1, 2\}$, $u_i(\sigma) = u_i(\mu)$.

Proof. First note that σ_2 may be written:

$$\sigma_2 = \sum_{s_1 \in S_1 : \mu(s_1 \times S_2) > 0} \mu(s_1 \times S_2) \mu(.|s_1) \tag{10}$$

Proof of (a): assume $\mu(s_1 \times S_2) > 0$; then by lemma 6.4 $\mu(.|s_1) \in NE_2$. Therefore, by (10) and convexity of NE_2, $\sigma_2 \in NE_2$. Similarly, $\sigma_1 \in NE_1$, so that, by proposition 6.3, $\sigma \in NE$.

Proof of (b): assume $\mu(s_1 \times S_2) > 0$; then s_1 is coherent and, by the analogous for player 2 of $(ii) \Rightarrow (iii)$ in lemma 6.4, any coherent strategy of player 1 is a best response to $\mu(.|s_1)$. Since $\sigma_1 \in NE_1$, σ_1 has support in S_1^c, so that

$$u_1(\sigma_1, \mu(.|s_1)) = u_1(s_1, \mu(.|s_1)) \tag{11}$$

Using successively (10), (11) and a straightforward computation, we get

$$\begin{aligned} u_1(\sigma) &= \textstyle\sum_{s_1 \in S_1 : \mu(s_1 \times S_2) > 0} \mu(s_1 \times S_2) u_1(\sigma_1, \mu(.|s_1)) \\ &= \textstyle\sum_{s_1 \in S_1 : \mu(s_1 \times S_2) > 0} \mu(s_1 \times S_2) u_1(s_1, \mu(.|s_1)) \quad = \quad u_1(\mu) \end{aligned}$$

Similarly, $u_2(\sigma) = u_2(\mu)$, completing the proof. ■

As mentioned in [8], if a two-player zero-sum game has a unique Nash equilibrium σ then $C = \{\sigma\}$. Similarly:

Corollary 6.6 *A two-player prebinding game has a unique Nash equilibrium if and only if it has a unique correlated equilibrium distribution.*

Proof. Let G be a two-player prebinding game. Assume that G has a unique Nash equilibrium σ. Necessarily, σ is an extreme Nash equilibrium (in the sense of [7]). But in two-player games, an extreme Nash equilibrium is an extreme point of C [7]. Therefore σ does not belong to the relative interior of C. Therefore, by theorem 5.3, C is a singleton. Conversely, if C is a singleton, G has trivially a unique Nash equilibrium. ■

6.3 Equilibrium payoffs of prebinding games

Let NEP (resp. NEP_i, CEP) denote the set of Nash equilibrium payoffs (resp. Nash equilibrium payoffs of player i, correlated equilibrium payoffs). That is,

$$NEP = \{g = (g_i)_{i\in I} \in \mathbb{R}^I \,/\, \exists\sigma \in NE, \forall i \in I, u_i(\sigma) = g_i\}$$

$$NEP_i = \{g_i \in \mathbb{R} \,/\, \exists\sigma \in NE, u_i(\sigma) = g_i\}$$

$$CEP = \{g = (g_i)_{i\in I} \in \mathbb{R}^I \,/\, \exists\mu \in C, \forall i \in I, u_i(\mu) = g_i\}$$

Two-player games which are best-response equivalent to zero-sum games may have an infinity of Nash equilibrium payoffs (for instance, see [23, example 5.20]). So prebinding games do not generally have a unique Nash equilibrium payoff. Nonetheless some of the properties of equilibrium payoffs of zero-sum games are preserved. In particular, proposition 6.3 and proposition 6.5 imply respectively that:

Corollary 6.7 *In a two-player prebinding game, NEP_1 and NEP_2 are convex and $NEP = NEP_1 \times NEP_2$*

Corollary 6.8 *In a two-player prebinding game, $CEP = NEP$*

Thus, allowing for correlation is useless in two-player prebinding games, in the sense that it cannot improve the payoffs of the players in equilibria. Furthermore:

Corollary 6.9 *In a two-player prebinding game, any correlated equilibrium distribution payoff of player i given his move is a Nash equilibrium payoff of player i:*

$$\forall\mu \in C, \forall i \in \{1,2\}, \forall s_i \in S_i, \mu(s_i \times S_{-i}) > 0 \Rightarrow \sum_{s_{-i}\in S_{-i}} \mu(s_{-i}|s_i)u_i(s) \in NEP_i$$

Proof. For clarity we take $i = 1$. In (11), $(\sigma_1, \mu(.|s_1))$ is a Nash equilibrium (by lemma 6.4, proposition 6.5(a) and proposition 6.3). Therefore, $u_1(s_1, \mu(.|s_1)) = \sum_{s_2\in S_2} \mu(s_2|s_1)u_1(s) \in NEP_1$. ■

6.4 Discussion

(a) Several classes of non-zero sum games in which some of the properties of two-player zero-sum games are still satisfied have been studied. Most are defined in either of these three ways:

(i) by requiring some conflict in the preferences of the players over strategy profiles ("Strictly competitive games" [1], [9], "Unilaterally competitive games" [15]);

(ii) by comparing the payoff structure in G and in some zero-sum game ("Strategically zero-sum games" [17], games "best-response equivalent" [21] or "order-equivalent" [22] to a zero-sum game);

(iii) by comparing the Nash equilibria or Nash equilibrium payoffs of G and of some auxiliary game ("Almost strictly competitive games" [1] and several other classes of games studied in [4]).

The definition of binding and prebinding games do not fall in these categories; binding games however may be defined by comparing the *correlated equilibria* of G and of some auxiliary game. Indeed, let $-G$ be the game with the same sets of players and strategies than G but in which all the payoffs are reversed:

$$-G = \{I, (S_i)_{i \in I}, (-u_i)_{i \in I}\}$$

We let the reader check that G is binding if and only if G and $-G$ have the same correlated equilibria.

(b) Lemma 6.4 implies that in two-player binding games, as in two-player zero-sum games, the Nash equilibrium strategies of the players can be computed independently, as solutions of linear programs that depend only on the payoffs of the *other* player. In two-player prebinding games, the additional knowledge of the sets of individually coherent strategies is required[14].

(c) A wide range of dynamic procedures converge towards correlated equilibrium distributions in all games (for instance generalized no-regret procedures [12], [13]). By proposition 6.5, suitably modified versions of these dynamics converge towards Nash equilibria in all two-player prebinding games.

(d) In 3-player binding games, Nash equilibria are not exchangeable (see [19, section 6]). To my knowledge, whether the other properties of section 6 extend to n-player games is open.

A Elements of Dual Reduction

We recall here some basic elements of dual reductions that are useful in the proofs (for more details, see [18] and [23]).

(a) Let $i \in I$. Consider a mapping

$$\begin{array}{rccc} \alpha_i : & S_i & \to & \Delta(S_i) \\ & s_i & \to & \alpha_i * s_i \end{array}$$

[14] Indeed the 1×2 games $\begin{pmatrix} 0,1 & 0,0 \end{pmatrix}$ and $\begin{pmatrix} 0,0 & 0,1 \end{pmatrix}$ are both prebinding and in both games the payoffs of player 1 are the same. However, the Nash equilibrium strategies of player 2 are not the same in both games.

That is, α_i associates to every element of S_i a probability distribution over S_i. This mapping induces a Markov chain on S_i. We denote by S_i/α_i a basis of the invariant measures on S_i for this Markov chain. A mixed strategy $\sigma_i \in \Delta(S_i)$ is α_i-invariant [in the sense that

$$\forall t_i \in S_i, \sum_{s_i \in S_i} \sigma(s_i)\, \alpha_i * s_i(t_i) = \sigma_i(t_i)\ ]$$

if and only if $\sigma_i \in \Delta(S_i/\alpha_i)$.

(b) Let $\alpha = (\alpha_i)_{i \in I}$ be a a vector of mappings $\alpha_i : S_i \to \Delta(S_i)$. The α-reduced game G/α is the game obtained from G by restricting the players to their α-invariant strategies. That is,

$$G/\alpha = \{I, (S_i/\alpha_i)_{i \in I}, (u_i)_{i \in I}\}$$

Let $s \in S$. Define:

$$g(\alpha, s) = \sum_{i \in I} [u_i(\alpha_i * s_i, s_{-i}) - u_i(s)] \tag{12}$$

Myerson [18] defines α to be a *dual vector* if $g(\alpha, s) \geq 0$ for all s in S. A dual vector is full if, for all (i, s_i, t_i) in $I \times S_i \times S_i$, $\alpha_i * s_i(t_i)$ is positive whenever t_i jeopardizes s_i. There exist full dual vectors. The set of dual vectors is convex and any positive convex combination of a dual vector with a full dual vector is a full dual vector. *A (full) dual reduction of G is an α-reduced game G/α where α is a (full) dual vector.* (The terminology is somewhat ambiguous as "dual reduction" may refer either to a reduced game or to the reduction technique.)

(c) Let $(i, s_i, t_i) \in I \times S_i \times S_i$. Generally (that is, whether α is a dual vector or not), if $\alpha_i * s_i(t_i)$ is positive then, in G/α, s_i is either eliminated or grouped with t_i. So, if α is a full dual vector: if s_i jeopardizes t_i, then in G/α, s_i is either eliminated or grouped with t_i; if s_i and t_i jeopardize each other then in G/α, s_i and t_i are either both eliminated or grouped together. Moreover, in full dual reductions, all incoherent strategies are eliminated.

(d) Let α be a dual vector. A probability distribution μ over $S/\alpha = \times_{i \in I} S_i/\alpha_i$ induces a probability distribution $\tilde{\mu}$ over S:

$$\tilde{\mu}(s) = \sum_{\sigma \in S/\alpha} \mu(\sigma)\sigma(s)$$

If μ is a correlated (resp. Nash) equilibrium distribution of G/α then $\tilde{\mu}$ is a correlated (resp. Nash) equilibrium distribution of G.

References

[1] Aumann R.J. (1961), "Almost Strictly Competitive Games", *Journal of the SIAM* **9**, 544-550

[2] Aumann, R.J. (1974), "Subjectivity and Correlation in Randomized Strategies", *Journal of Mathematical Economics* **1**, 67-96

[3] Aumann, R.J. (1987), “Correlated Equilibria as an Expression of Bayesian Rationality”, *Econometrica* **55**, 67-96

[4] Beaud, J.P. (2002), “Contributions à la théorie des jeux : jeux antagonistes et modèles de jeux répétés”, Ph’D thesis, Université Paris VI.

[5] Bernheim D.G. (1984), “Rationalizable Strategic Behavior”, *Econometrica* **52**, 1007-1028

[6] Cripps, M. (1991), “Correlated Equilibria and Evolutionnary Stability”, *Journal of Economic Theory* **55**, 428-434

[7] Evangelista, F. and T.E.S. Raghavan (1996), “A note on correlated equilibrium”, *International Journal of Game Theory* **25**, 35-41

[8] Forges, F. (1990), “Correlated Equilibrium in Two-Person Zero-Sum Games”, *Econometrica* **58**, 515

[9] Friedman J. (1983), “On Characterizing Equilibrium Points in Two-Person Strictly Competitive Games”, *International Journal of Game Theory* **12**, 245-247.

[10] Gilboa I. and E. Zemel (1989), “Nash and Correlated Equilibria: Some Complexity Considerations”, *Games and Economic Behavior* **1**, 80-93

[11] Gomez Canovas, S., P. Hansen, and B. Jaumard (1999), “Nash Equilibria from the Correlated Equilibria Viewpoint”, *International Game Theory Review* **1**, 33-44

[12] Hart, S. and A. Mas-Colell (2001), “A General Class of Adaptive Strategies”, *Journal of Economic Theory* **98**, 26-54

[13] Hart, S. and A. Mas-Colell (2003), “Regret-Based Continuous-Time Dynamics”, *Games and Economic Behavior* **45**, 375-394

[14] Hart, S., and D. Schmeidler (1989), “Existence of Correlated Equilibria”, *Mathematics Of Operations Research* **14**, 18-25

[15] Kats, A. and J.-F. Thisse (1992), “Unilateraly Competitive Games”, *International Journal of Game Theory* **21**, 291-299

[16] Maynard Smith, J., and G.A. Parker (1976), “The Logic of Asymmetric Contests”, *Animal Behavior* **24**, 159-175

[17] Moulin, H and J-P. Vial (1978), “Strategically Zero-Sum Games: The Class of Games Whose Completely Mixed Equilibria Cannot be Improved Upon”, *International Journal of Game Theory* **7**, 201-221

[18] Myerson, R.B. (1997), “Dual Reduction and Elementary Games”, *Games and Economic Behavior* **21**, 183-202

[19] Nau R.F., S. Gomez Canovas, and P. Hansen (2003), “On the Geometry of Nash and Correlated Equilibria”, working paper, Fuqua School of Business, Duke University

[20] Nau, R.F. and K.F. McCardle (1990), "Coherent Behavior in Noncooperative Games", *Journal of Economic Theory* **50**, 424-444

[21] Rosenthal, R.W. (1974), "Correlated Equilibria in Some Classes of Two-Person Games", *International Journal of Game Theory* **3**, 119-128

[22] Shapley, L. (1964), "Some Topics in Two-Person Games", *Annals of Mathematical Studies* **52**, Princeton, 1-28

[23] Viossat, Y. (2003), "Properties of Dual Reduction", cahier du laboratoire d'économétrie, Ecole polytechnique, Paris